Game Theory with Computer Science Applications
Lecture 4: LP Duality and Zero-Sum Game

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Linear Programming (LP)

$$\begin{aligned} max_x \quad & c^T x \\ s.t., \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

- The above linear programming also calls primal problem.
- Interpretation from economics.
  - x_j: amount of product j produced.
  - c_j: profit from 1 unit of product j.
  - b_i: amount of raw materials of type i available.
  - $a_{i,j}$: amount of raw materials i used to produce 1 unit of product j.
- Thus, the LP is the problem of maximizing profit subject to resource constraints.

Duality of LP

$$\begin{aligned} min_y \quad & b^T y \\ s.t., \quad & A^T y \geq c \\ & y \geq 0 \end{aligned}$$

- How much would someone be willing to pay for each unit of raw material i ? If the constraints are not satisfied, the seller would not sell the products.
- Intuitively, $c^T x^* = b^T y^*$, i.e., the optimal objectives of the primal and dual are equal.
- Otherwise, the seller won't sell or buyer won't buy.

Strong Duality Theorem for LPs

Strong Duality

If the primal and dual of LPs are **feasible**, then both have the same optimal objective value.

To prove strong duality, we need the following result called Farkas's lemma.

Farkas's Lemma

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^{m \times 1}$, one of the following statements is true:
- (1) $\exists x \geq 0$, s.t., $Ax = b$;
- (2) $\exists y$: $y^T A \geq 0$ and $y^T b < 0$.

Farkas' Lemma is an example of a theorem of alternative. It states that exactly one of two statements is true, but not both.

Geometric interpretation of Farkas lemma

- The geometric interpretation of Farkas lemma illustrates the connection to the **separating hyperplane theorem**.
- Let a_1 and a_2 be the columns of A. Define

$$Q = cone(a_1, a_2) = \{s \in R^2 : s = \lambda_1 a_1 + \lambda_2 a_2, \lambda_1 \geq 0, \lambda_2 \geq 0\}.$$

- If $b \notin cone(a_1, a_2)$, then we can separate it from the cone with a hyperplane.
- The geometric interpretation of Farkas lemma is either b is in the $cone(a_1, a_2)$, or we can find a hyperplane to separate b and $cone(a_1, a_2)$.

Proof of LP duality

- First, we will prove that if x is feasible for the primal and y is feasible for the dual, then we have $c^T x \leq b^T y$.
- To see this, we have

$$
\begin{aligned}
c^T x &= x^T c \\
\Rightarrow \quad c^T x &\leq x^T A^T y \\
\Rightarrow \quad c^T x &\leq by
\end{aligned}
$$

The first inequality comes from $A^T y \geq c$ and $x \geq 0$, and the second inequality comes from $Ax \leq b$ and $y \geq 0$.

Proof of LP duality (continued)

- We will next prove that $c^T x^* \geq b^T y^*$ for optimal x^* (primal) and y^* (dual).
- Let $c^T x^* = \Delta$. This implies that $\nexists x$, s.t.,

$$\begin{aligned} c^T x &\geq \Delta + \epsilon, \quad \text{(for any } \epsilon > 0) \\ Ax &\leq b, \\ x &\geq 0. \end{aligned}$$

$$\Leftrightarrow \quad \begin{aligned} -c^T x + \gamma_0 &= -\Delta - \epsilon, \\ Ax + \gamma &= b \\ \gamma_0, \gamma, x &\geq 0. \end{aligned}$$

$$\Leftrightarrow \quad \begin{pmatrix} -c^T & 1 & 0 \\ A & 0 & I \end{pmatrix} \begin{pmatrix} x \\ \gamma_0 \\ \gamma \end{pmatrix} = \begin{pmatrix} -\Delta - \epsilon \\ b \end{pmatrix}$$

Proof of LP duality (continued)

- By Farkas' Lemma, $\exists \lambda_0, \lambda_1$:

$$-c^T\lambda_0 + \lambda_1^T A \geq 0, \quad \lambda_0 \geq 0, \quad \lambda_1^T \geq 0.$$

  and $-(\Delta+\epsilon)\lambda_0 + \lambda_1 b < 0$.
- We can have

$$\frac{\lambda_1^T}{\lambda_0} A \geq c^T, \quad \frac{\lambda_1^T}{\lambda_0} \geq 0, \quad \frac{\lambda_1^T}{\lambda_0} b < \Delta + \epsilon.$$

  Letting $y = \frac{\lambda_1^T}{\lambda_0}$, that is $\exists y$, s.t.,

$$\begin{array}{rcl} y^T b & < & \Delta + \epsilon \\ y^T A & \geq & c^T, \\ y & \geq & 0, \end{array}$$

  Thus, for each $\epsilon > 0$, $\exists$ dual feasible y, s.t., $y^T b < \Delta + \epsilon$. Since the dual is a min problem, we can have dual optimal objective $< \Delta + \epsilon, \forall \epsilon > 0$, that is dual optimal objective $\leq \Delta$.

Remarks for LP duality

- Dual can also be defined for problems with a mixture of equality and inequality constraints, as well as some variables are unconstrained.
- Another Interpretation of LP duality.

$$\begin{aligned} max_x \quad & c^T x \\ s.t., \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

can always be written as

$$\begin{aligned} max_x \quad & c^T x \\ s.t., \quad & a_1^T x \leq b_1 \\ & \vdots \\ & a_n^T x \leq b_n \\ & x \geq 0, \end{aligned}$$

where a_i's are the columns of A.

Remarks for LP duality (continued)

Suppose we want to get an upper bound on the optimal objective of the above LP. We can proceed as follows. Let $y_1, \cdots, y_n \geq 0$. Then

$$(\sum_i y_i a_i^T)x \leq \sum_i b_i y_i.$$

If $\sum_i y_i a_i^T \geq c^T$(element wise), then gives an upper bound to the objective,

$$c^T x \leq (\sum_i y_i a_i^T)x \leq \sum_i b_i y_i = b^T y.$$

Thus, the highest upper bound is obtained by solving

$$\begin{aligned} min_y \quad & b^T y \\ s.t., \quad & \sum_i a_i y_i \geq c \quad (or \quad A^T y \geq c) \\ & y \geq 0. \end{aligned}$$

This is the dual. LP duality says that this is not just an upper bound, but in fact equal to the optimal objective of the original LP.

Zero-sum Game

- Two players P1 and P2.
- $A(i,j)$: Utility to P1 when P1 takes action i and P2 takes action j.
- Mixed Strategy: x is probability distribution over P1's actions; y is probability distribution over P2's actions.
- Utility to P1 when P1 takes strategy x and P2 takes strategy y.

$$U_1(x,y) = \sum_{i,j} A(i,j) \times x_i \times y_j = x^T A y.$$

From zero-sum game, we can have $U_2(x,y) = -U_1(x,y)$.

- A NE (x^*, y^*) satisfies

$$x^{*T} A y^* \geq x^{*T} A y^* \quad \forall x,$$

$$\text{and} \quad x^{*T} A y^* \leq x^{*T} A y \quad \forall y,$$

That is (x^*, y^*) constitutes a saddle point.

Minimax Theorem

Minimax Theorem (Von Neumann)

The following three statements are true.

(i) $min_y max_x x^T Ay = max_x min_y x^T Ay$ This is called the value of the game.

(ii) Suppose x^* solves

$$max_x(min_y x^T Ay) \tag{1}$$

and y^* solves

$$min_x(max_y x^T Ay) \tag{2}$$

${x^*}^T Ay^*$ is the value of the game, and (x^*, y^*) constitutes a NE.

(iii) If (x^*, y^*) is a NE, then ${x^*}^T Ay^*$ is the value of the game, and x^* solves (1), and y^* solves (2).

Proof for MiniMax Theorem (i)

- Suppose wants to maximize its worst-case payoff

$$max_x min_y x^T Ay = max_x min_j (x^T A).$$

The above equality comes from that $y_j \geq 0$ and $\sum_j y_j = 1$.

- This is equivalent to

$$\begin{aligned} max_{x,v_1} \quad & v_1 \qquad (LP1) \\ s.t., \quad & v_1 \leq (x^T A)_j \quad \forall j, \quad \text{and} \quad x \geq 0. \end{aligned}$$

Similarly, P2's problem is

$$\begin{aligned} max_{y,v_2} \quad & v_2 \qquad (LP2) \\ s.t., \quad & v_2 \geq (Ay)_i \quad \forall i, \quad \text{and} \quad y \geq 0. \end{aligned}$$

It can be verified that problem (LP2) is the dual of (LP1). Thus, by strong duality, we have $v_1^* = v_2^*$ or

$$max_x min_y x^T Ay = min_y max_x x^T Ay.$$

Proof for MiniMax Theorem (ii)

Let x^* be a solution to LP1. From the constraints

$$\begin{array}{rcl} & v_1^* & \leq (x^{*T}A)_j \quad \forall j \\ \Rightarrow & \sum_j v_1^* \times y_j^* & \leq \sum_j (x^{*T}A)_j \times y_j^* \\ \Rightarrow & v_1^* & \leq (x^{*T}A)y^* \end{array}$$

where y^* is a solution to LP2. Similarly, we can show that

$$v_2^* \geq x^{*T}Ay^*.$$

Since $v_1^* = v_2^*$, we have

$$x^{*T}Ay^* = min_y max_x x^T Ay = max_x min_y x^T Ay$$

Proof for MiniMax Theorem (ii) (continued)

Recall that x^* solves

$$max_x(min_y x^T Ay),$$

and y^* solves

$$min_y(max_x x^T Ay),$$

and by (i), we have $maxmin = minmax$. Thus,

$$\begin{aligned} min_y {x^*}^T Ay &= max_x min_y x^T Ay = min_y max_x x^T Ay \\ &= max_x x^T Ay^* \\ &\geq {x^*}^T Ay^* \end{aligned}$$

$\Rightarrow$ y^* solves $min_y {x^*}^T Ay$. Similarly, we can show that x^* solves $max_x x^T Ay^*$.
$\Rightarrow$ (x^*, y^*) is a SP.

Proof for MiniMax Theorem (iii)

Since (x^*, y^*) is a NE, we have

$$x^{*T}Ay^* = min_y x^{*T}Ay \leq max_x min_y x^T Ay.$$

Similarly, we have

$$x^{*T}Ay^* = max_x xAy^* \geq min_y max_x x^T Ay.$$

We then have

$$min_y max_x x^T Ay \leq x^{*T}Ay^* \leq max_x min_y x^T Ay. \quad (3)$$

We also have

$$min \quad max \geq max \quad min.$$

Thus, equality holds throughout in (3) above.

Proof for MiniMax Theorem (iii) (continued)

Note that since (x^*, y^*) is a NE, we have

$$\begin{aligned} min_y {x^*}^T Ay &= {x^*}^T Ay^* \\ &= max_x min_y x^T Ay \\ &\geq min_y x^T Ay, \quad \forall x. \end{aligned}$$

We thus have x^* maximizes $min_y(x^T Ay)$.
Similarly, y^* maximizes $max_x(x^T Ay)$.