Game Theory with Computer Science Applications
Lecture 3: Existence of a Nash Equilibrium

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Game Setting and related Concepts

- Pure Strategy for agent i: $x_i \in X_i$ (discrete, finite set).
- Mixed Strategy for agent i: $p_i(x_i) = \Pr(\text{agent i plays action } x_i)$.
- Utility to i: $U_i(x_i, \mathbf{x}_{-i})$ and $U_i(p_i, \mathbf{p}_{-i})$.
- Some concepts: closed set, bounded set, convex set, continuous functions.

The Nash's Theorem

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Any finite strategic game has a mixed strategy Nash Equilibrium.

Brouwer Fixed Point Theorem

Brouwer Fixed Point Theorem

Let $C \subseteq \mathbb{R}^n$ be a **compact (closed and bounded)** and **convex** set. Let $f\colon C \to C$ be a continuous function. Then f has a fixed pointed in C, *i.e.*, $x \in C$, *s.t.*, $x = f(x)$.

Proof.

For the one-dimensional case. When $n = 1$, the convex and compact sets are closed intervals $[a, b]$. Let $f\colon [a, b] \leftarrow [a, b]$. If $f(a) = a$ or $f(b) = b$ we are done. Suppose $f(a) > a$ and $f(b) < b$. Consider $g(x) = f(x) - x$. Then $g(a) > 0$, $g(b) < 0$. g is continuous because f is continuous. The ***intermediate value theorem*** tells us that there is some $a < x^* < b$, such that $g(x^*) = 0$. □

Proof for Nash Theorem using Brouwer FP Theorem

- Let $(p_1, p_2, \cdots, p_n)$ be a set of strategies.
- Define $r_i(x_i) = (U_i(x_i, p_{-i}) - U_i(p_i, p_{-i}))^+$, i.e., $r_i(x_i)$ is the amount by which the expected utility to i can be increased by changing strategy from p_i to x_i.
- Define

$$f_i(p_i(x_i)) = \frac{p_i(x_i) + r_i(x_i)}{\sum_x (p_i(x) + r_i(x))}.$$

- $(p_1, p_2, \cdots, p_n)$ is a **convex and compact set**.
  $f(p) = (f_1(p_1), f_2(p_2), \cdots, f_n(p_n))$ is a **continuous function**.
  Homework.
- We then have a fixed point p:

$$p_i(x_i) = \frac{p_i(x_i) + r_i(x_i)}{\sum_x (p_i(x) + r_i(x))}.$$

Proof for Nash Theorem using Brouwer (continued)

- We will now show that for such a fixed point,

$$r_i(x_i) = 0 \quad \forall i, x_i.$$

  i.e., no increase in utility is possible by changing strategy from p_i to x_i. Thus, such a fixed point is a NE.

- First, we claim that **for each** i**,** $\exists x_i$**, s.t.,** $r_i(x_i) = 0$**.** We will prove this by contradiction. Suppose for some i, $r_i(x_i) > 0$, $\forall x_i$. Then,

$$\begin{aligned} & U_i(x_i, p_{-i}) > U_i(p_i, p_{-i}), \quad \forall x_i. \\ \Rightarrow \quad & \sum_{x_i} p_i(x_i) U_i(x_i, p_{-i}) > U_i(p_i, p_{-i}), \\ \Rightarrow \quad & U_i(p_i, p_{-i}) > U_i(p_i, p_{-i}), \end{aligned}$$

  which is a contradiction.

Proof for Nash Theorem using Brouwer (continued)

- Fix i, let x_i be s.t., $r_i(x_i) = 0$. Then,

$$
\begin{aligned}
& p_i(x_i) = \frac{p_i(x_i)}{\sum_x (p_i(x) + r_i(x))} \\
\Rightarrow \quad & \sum_x p_i(x) + \sum_x r_i(x) = 1 \\
\Rightarrow \quad & \sum_x r_i(x) = 0 \\
\Rightarrow \quad & r_i(x) = 0, \forall x \in A_i.
\end{aligned}
$$

This completes the proof.

Kakutani Fixed-point Theorem

Kakutani Fixed-point Theorem

Let C be a convex and compact subset of $\mathbb{R}^n$. Let f be a correspondence mapping each point in C to a subset of a C, i.e., $f: C \to 2^C$. Suppose the following three conditions hold:

- $f(x) \neq \emptyset, \quad \forall x,$
- $f(x)$ is a convex set, $\forall x,$
- f has a closed graph: if $\{x_n, y_n\} \to \{x, y\}$ with $y_n \in f(x_n)$, then $y \in f(x)$.

Then f has a fixed point in C.

Weierstrass's Theorem

Let A be a non-empty, compact subset of $\mathbb{R}^n$ and let $f: A \to \mathbb{R}$ be a continuous function. Then, there exists an optimal solution to the optimization problem $Min\ f(x), \quad x \in A$.

Proof for Nash Theorem using Kakutani

- We will apply Kakutani's fixed point theorem to establish the existence of a solution to

$$p \in BP(p),$$

where $p = (p_1, p_2, \cdots, p_n)$ and $BP(p) = (BP(p_{-1}), \cdots, BP(p_{-n}))$.

- We will verify the mapping BP satisfies the conditions required in the Kakutani's fixed-point theorem.
- (1) $BP(p)$ **is a non-empty set for each** p**.**
  This is because $\max_{p_i \in \Delta X_i} U_i(p_i, p_{-i})$ is a maximization problem of a continuous function over the set of the probability distribution over X_i, which is a compact set. The result follows from **Weierstrass' extreme value theorem**.

Proof for Nash Theorem using Kakutani (continued)

- (2) **For each p, $BP(p)$ is a convex set.**
  We recall that

$$U_i(p_i, p_{-i}) = \sum_x p_1(x_1) \times \cdots \times p_n(x_n) \times U_i(x_i, x_{-i}).$$

So if $p_i^*, \tilde{p}_i \in BP(p_{-i})$, as $U_i(p_i^*, p_{-i}) = U_i(\tilde{p}_i, p_{-i})$, we can verify that

$$U_i(\alpha \times p_i^* + (1-\alpha) \times \tilde{p}_i, p_{-i}) = U_i(p_i^*, p_{-i}), \quad \forall \alpha \in [0,1].$$

Hence, we have $\alpha \times p_i^* + (1-\alpha) \times \tilde{p}_i \in BP(p_{-i})$

Proof for Nash Theorem using Kakutani (continued)

- (3) **We will now show that BP has a closed graph.** Let $(p_i^n, p_{-i}^n) \to (p_i, p_{-i})$ with $p_i^n \in BP(p_{-i}^n)$. Suppose that $p_i \notin BP(p_{-i})$. Then $\exists \tilde{p}_i$ and $\epsilon > 0$ s.t.,

$$U_i(\tilde{p}_i, p_{-i}) \geq U_i(p_i, p_{-i}) + \epsilon.$$

- We next show that $\tilde{p}_i$ is a better response for p_{-i}^n (for some n) than p_i^n, and thus contradicts $p_i^n \in BP(p_{-i}^n)$.
- For sufficiently large n,

$$\begin{aligned} U_i(\tilde{p}_i, p_{-i}^n) &\geq U_i(\tilde{p}_i, p_{-i}) - \frac{\epsilon}{2} && (1) \\ &\geq U_i(p_i, p_{-i}) + \epsilon - \frac{\epsilon}{2} && (2) \\ &\geq U_i(p_i^n, p_{-i}^n) - \frac{\epsilon}{4} + \frac{\epsilon}{2} && (3) \\ &= U_i(p_i^n, p_{-i}^n) + \frac{\epsilon}{4}. && (4) \end{aligned}$$

Proof for Nash Theorem using Kakutani (continued)

- (1) comes from that $p^n_{-i} \to p_{-i}$ and U_i is continuous.
  (3) comes from that for sufficiently large n, $(p^n_i, p^n_{-i}) \to (p_i, p_{-i})$ and U_i is continuous.
- The above result contradicts $p^n_i \in BP(p^n_{-i})$. Thus, BP has a closed graph.
- Nash's Theorem follows from the Kakutani fixed point theorem.

Games with infinite strategies

- N agents.
- Strategy in $x_i \in X_i \subseteq \mathbb{R}^{n_i}$, X_i contains typically an uncountable number of points.
- Utility/Payoff to agent i: $u_i(x_i, \mathbf{x}_{-i})$.
- Two types of constraints would be imposed on strategy profile $\mathbf{x}$.
- **Coupled constraints**:

$$\mathbf{x} \in \Omega \subseteq \mathbb{R}^{n_1+n_2+\cdots+n_N}.$$

  E.g., $N = 2$, $3 \times x_1 + 2 \times x_2 \leq 6.$, and $x_1 \geq 0$, $x_2 \geq 0$. Here, the constraints on x_1 and x_2 are coupled, i.e., if $x_1 = 1$, then if it results $0 \leq x_2 \leq 3/2$.
- **Uncoupled constraints**:

$$x_i \in X_i \subseteq \mathbb{R}^{n_i}, \text{ and } \Omega = X_1 \times X_2 \times X_n.$$

  E.g., N=2, $0 \leq x_1 \leq 2$ and $0 \leq x_2 \leq 3$. The choice of x_i does not affect the constraints on x_{-i}.

Existence of Nash Equilibrium for Infinite Games

Glicksberg Theorem

Consider the uncoupled constraint, i.e., $x_i \in X_i$ and $\Omega = X_1 \times X_2 \cdots X_N$.
Suppose
- each X_i is non-empty and compact,
- and that $u_i(x_i, \mathbf{x}_{-i})$ is continuous on Ω.

Then, there exists a mixed strategy NE for this game.

Proof.

Intuition behind the proof:
- Discrete the strategy space, and consider the resulting finite-strategy game.
- By Nash's Theorem, a mixed Nash Equilibrium (NE) exists for the discrete game.
- Show that as the discretization becomes finer and finer, the NE converges to a NE of the continuous games. □

Conditions for Existence of Pure NE

Rosen's Theorem

Let Ω be a coupled constraint set. Assume that
- Ω is a convex and compact set,
- and that $u_i(x_i, \mathbf{x}_{-i})$ is continuous on Ω.
- Further, suppose that $u_i(x_i, \mathbf{x}_{-i})$ is concave in x_i for each $\mathbf{x}_{-i}$.

Then, there exist a pure NE.

Proof: Consider the function defined over $\Omega \times \Omega$:

$$L(\mathbf{v}, \mathbf{x}) = \sum_{i=1}^{N} u_i(v_i, \mathbf{x}_{-i}).$$

We first note that if there exists a strategy profile $\mathbf{x}$ s.t.,

$$L(\mathbf{x}, \mathbf{x}) \geq L(\mathbf{v}, \mathbf{x}), \forall v \in \Omega. \tag{5}$$

Then $\mathbf{x}$ must be a NE. We can see this by proving a contradiction.

Proof (continued)

Suppose $\mathbf{x}$ satisfies (5), but is not a NE, i.e., $\exists i$ s.t.,

$$u_i(v_i, \mathbf{x}_{-i}) > u_i(x_i, \mathbf{x}_{-i}), \quad \forall (v_i, \mathbf{x}_{-i}) \in \Omega.$$

Then

$$\begin{aligned} & u_i(v_i, \mathbf{x}_{-i}) + \sum_{j \neq i} u_j(x_j, \mathbf{x}_{-j}) > u_i(x_i, \mathbf{x}_{-i}) + \sum_{j \neq i} u_j(x_j, \mathbf{x}_{-j}) \\ \Rightarrow \quad & L(\widehat{\mathbf{v}}, \mathbf{x}) > L(\mathbf{x}, \mathbf{x}), \end{aligned}$$

where $\widehat{\mathbf{v}} = (x_1, \cdots, x_{i-1}, v_i, x_{i+1}, \cdots, x_N) \in \Omega$. This contradicts to (5).
We next show that if there exists a strategy profile $\mathbf{x}$ s.t.,

$$\mathbf{x} \in argmax_{v \in \Omega} L(\mathbf{v}, \mathbf{x}),$$

then such a $\mathbf{x}$ satisfies (5), and hence would be a NE.

Proof (continued)

- Let $g(\mathbf{x}) = argmax_{v \in \Omega} L(\mathbf{v}, \mathbf{x})$. We note that $g(\mathbf{x})$ is a set, and thus is a correspondence, not a function.
- We now show that $g(\mathbf{x})$ has a fixed point, i.e., $\exists \mathbf{x}$, s.t., $\mathbf{x} \in g(\mathbf{x})$. We prove this by using the Kakutani's fixed point theorem.
- Ω is assumed to be **convex** and **compact** as required by Kakutani's theorem.
- For each $\mathbf{x} \in \Omega$, $g(\mathbf{x})$ is **non-empty and convex.**
  - $g(\mathbf{x})$ is non-empty comes from that $L(\mathbf{v}, \mathbf{x})$ is a continuous function of $\mathbf{v}$ and Ω is a compact set. Continuous functions over compact sets have a max by Weirstrass's theorem.
  - $L(\mathbf{v}, \mathbf{x})$ is a sum of concave functions, and thus is concave in $\mathbf{v}$. Let $\mathbf{v^1}$ and $\mathbf{v^2}$ be two elements of $g(\mathbf{x})$. And we have

  $$L(\alpha \times \mathbf{v^1} + (1-\alpha) \times \mathbf{v^2}, \mathbf{x}) \geq \alpha \times L(\mathbf{v^1}, \mathbf{x}) + (1-\alpha) \times L(\mathbf{v^2}, \mathbf{x}), \forall \alpha \in [0, 1].$$

  Thus, $\alpha \times \mathbf{v^1} + (1 - \alpha) \times \mathbf{v^2}$ also maximizes $L(\mathbf{v}, \mathbf{x})$ over $\mathbf{x} \in \Omega$. And $g(\mathbf{x})$ is a convex set.

Proof (continued)

- $g(\mathbf{x})$ is a closed graph can be established as in Nash's theorem.
- With the above three conditions, we can apply Kakutani's fixed point theorem.
- Thus, $g(\mathbf{x})$ has a fixed point, i.e., there exist an $\mathbf{x}$, s.t., $\mathbf{x} \in g(\mathbf{x})$.
- Such an $\mathbf{x}$ is a NE for this game.

Remark on Rosen's Theorem

- What is the reason to introduce the function $L(\mathbf{v}, \mathbf{x})$?
- Instead, suppose we follow the best responses' proof in Nash's Theorem. That is $g_1(\mathbf{x}_{-1})$ the BP of agent 1,

$$g_1(\mathbf{x}_{-1}) \in argmax_{(x_1, \mathbf{x}_{-1}) \in \Omega} u_1(x_1, \mathbf{x}_{-1}).$$

Similarly,

$$g_2(\mathbf{x}_{-2}) \in argmax_{(x_2, \mathbf{x}_{-2}) \in \Omega} u_2(x_2, \mathbf{x}_{-2}).$$

and so on.
For simplicity, there are two agents, i.e., $N = 2$. In this case, $\mathbf{x}_{-1} = x_2$ and $\mathbf{x}_{-2} = x_1$. We know that

$$(g_1(x_2), x_2) \in \Omega \quad \forall g_1(x_2).$$

$$(x_1, g_2(x_1)) \in \Omega \quad \forall g_2(x_1).$$

But if we consider the mapping, $\mathbf{x} \to (g_1(x_2), g_2(x_1)) \in \Omega$ It is not clear if $(g_1(x_2), g_2(x_1)) \in \Omega$. So, it is not obvious how to apply the FP theorem here.