Game Theory with Computer Science Applications
Lecture 2: Strategic Form Game, Equilibrium Concepts

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What is in a game?

- Players: Who?
- Strategies: What actions are available?
- Game rules: How? When? What?
- Outcomes: What results?
- Utilities: How do players evaluate outcomes of game?

Example:

- Players: Chess masters
- Strategies: Moving a piece
- Game rules: How pieces are moved
- Outcomes: Victory or defeat
- Utilities: Thrill of victory, agony of defeat

Rationality: Players are rational and self-interested, i.e., choose actions that maximize their utilities, given the available information.

Static games of complete information

- **Static games**: one-shot games, simultaneous-move games, e.g., rock-paper-scissors games.
- **Complete Information**:
  - All players know the structure of the game: players, strategies, game rules, outcomes, utilities.
  - All players know all players know the structure, ... and so on.
  - The structure of the game is common knowledge.

Strategic-form games

- Players: n agents
- (Pure) Strategies for agent i: action x_i, $x_i \in X_i$, X_i is a discrete finite set.
- Game rules: All agents simultaneously pick a strategy
- Outcomes: the strategy profile $\boldsymbol{x} = (x_1, \cdots, x_n)$. For convenience, we will also define $\boldsymbol{x} = (x_i, \boldsymbol{x}_{-i})$, where $\boldsymbol{x}_{-i} = (x_1, x_2, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$.
- Utilities: $u_i(x_i, \boldsymbol{x}_{-i})$ when agent i plays action x_i.

Mixed Strategies

Idea: an agent can randomize over pure strategies.

- **Pure Strategies** for agent i: X_i is a discrete finite set.
- **Mixed strategy** for agent i: p_i is the probability distribution over X_i.
  For $x \in X_i$, $p_i(x) = Pr(\text{agent i plays action x})$.
  $\Delta(X_i)$ denote the set of all probability distributions on X_i.
- Outcomes: the (mixed) strategy profile $\boldsymbol{p} = (p_1, \cdots, p_n) = (p_i, \boldsymbol{p}_{-i})$, where $\boldsymbol{p}_{-i} = (p_1, p_2, \cdots, p_{i-1}, p_{i+1}, \cdots, p_n)$.
- **Expected Utility**: to agent i under the mixed strategy profile $\boldsymbol{p} = (p_i, \boldsymbol{p}_{-i})$ is

$$U_i(p_i, \boldsymbol{p}_{-i}) = \sum_{x_1 \in X_1} \sum_{x_2 \in X_2} \cdots \sum_{x_n \in X_n} u_i(x_1, x_2, \cdots, x_n) p_1(x_1) \cdots p_n(x_n).$$

Best-Response Strategy

- Given p_{-i}, the **best-response strategy** for agent i is

$$BR_i(\boldsymbol{p}_{-i}) = argmax_{p_i} U_i(p_i, \boldsymbol{p}_{-i}).$$

- $BR_i(\boldsymbol{p}_{-i})$ is a set and it will include point mass probability measures corresponding to the best response actions for agent i given $\boldsymbol{p}_{-i}$.
- In fact, $BR_i(\boldsymbol{p}_{-i})$ consists of convex combinations of these best response actions. So, $BR_i(\boldsymbol{p}_{-i})$ is a convex set.
- $BP_i(\boldsymbol{p}_{-i})$ can be constructed as follows.
  (a) Find all pure strategy best responses to $\boldsymbol{p}_{-i}$; call this set $S_i(\boldsymbol{p}_{-i}) \subseteq X_i$
  (b) $BP_i(\boldsymbol{p}_{-i})$ is the set of all probability distributions over S_i, i.e., $BP_i(\boldsymbol{p}_{-i}) = \Delta(S_i(\boldsymbol{p}_{-i}))$.

Nash Equilibrium

Nash Equilibrium

$p^* = (p_1^*, \cdots, p_n^*)$ is a Nash Equilibrium (NE) if

$$U_i(p_i^*, p_{-i}^*) \geq U_i(p_i, p_{-i}^*), \quad \forall p_i.$$

- No agent can profitably deviate given the strategies of the other agents. Thus in NE, "best response correspondences intersect".
- Note that if the vector "equation"

$$\begin{pmatrix} p_1^* \\ \vdots \\ p_n^* \end{pmatrix} \in \begin{pmatrix} BP_1(p_{-1}^*) \\ \vdots \\ BP_n(p_{-n}^*) \end{pmatrix}$$

has a fixed point, i.e., $p^* = BP(p^*)$, then such as solution is a NE.

Example of matrix games: Partnership Game

- Partnership Game

P1 / P2	Work Hard	Be Lazy
Work Hard	(10,10)	(-5, 5)
Be Lazy	(5,-5)	(0,0)

- Two Pure NEs: (W, W) or (L, L). One of them is better than the other in terms of utilities to both agents.
- Do any other mixed NE exists?
  P1 plays W w.p. x and L w.p. $1 - x$.
  P2 plays W w.p. y and L w.p. $1 - y$.
- Utility to P1: 10 xy -5x(1-y) + 5(1-x) y, and Best Response for P1:

$$\begin{aligned} & max_{x\in[0,1]} 10xy - 5x(1-y) + 5(1-x)y, \\ = \quad & max_{x\in[0,1]} 5x(2y-1) + 5y \end{aligned}$$

Partnership Game (continued)

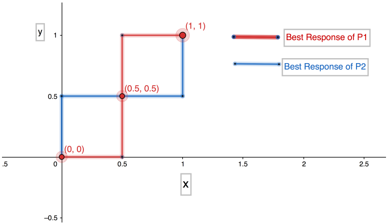
- Solving the above optimization, we can obtain best response for P1.

$$\text{if } y = \frac{1}{2}, x^* \in [0,1]; \quad y > \frac{1}{2}, x^* = 1; \quad y < \frac{1}{2}, x^* = 0.$$

- Best response for P2.

$$\text{if } x = \frac{1}{2}, y^* \in [0,1]; \quad x > \frac{1}{2}, y^* = 1; \quad x < \frac{1}{2}, y^* = 0.$$

- Visualize best response strategies of two players.



- There is a mixed NE at $x^* = \frac{1}{2}$, $y^* = \frac{1}{2}$. The utilities for both players are $(\frac{5}{2}, \frac{5}{2})$.

Example of matrix games: Battle of the sexes

- Battle of the sexes

Man/Woman	Basketball	Football
Basketball	(1,2)	(0, 0)
Football	(0,0)	(2,1)

- Two Pure NEs: (B, B) and (F, F).
- Does there exist mixed NE strategy?
  x= P(man plays B)
  y= P(woman plays B)
- Utility to Man: xy +2(1-x)(1-y) = x (3y-2) + 2(1-y) , and Best Response for Man:

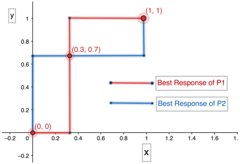
$$\begin{cases} x^* \in [0,1], & \text{if} \quad y = 2/3, \\ x^* = 0, & \text{if} \quad y < 2/3, \\ x^* = 1, & \text{if} \quad y > 2/3 \end{cases}$$

Battle of the sexes (continued)

- Similarly for the woman,

$$\begin{cases} y^* \in [0,1] & \text{if} \quad x = 1/3, \\ y^* = 0 & \text{if} \quad x < 1/3, \\ y^* = 1 & \text{if} \quad x > 1/3 \end{cases}$$

- Visualize best response strategies of two players.



- $x^* = \frac{1}{3}$ and $y^* = \frac{2}{3}$ is a mixed NE. The utilities for both players are $\frac{2}{3}$.

Matching Pennies

- Utility Matrix:

P1/P2	Head	Tail
Head	$(+1, -1)$	$(-1, +1)$
Tail	$(-1, +1)$	$(+1, -1)$

- No pure strategy NE.
- x= P(p1 uses H); y = P(p2 uses H).
- Utility to p1:
  $xy - x(1-y) - (1-x)y + (1-x)(1-y) = x(4y-2) + \cdots$

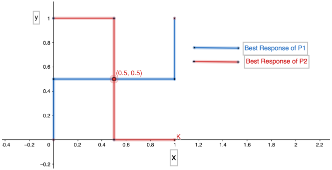
$$\Rightarrow \quad x^* = \begin{cases} [0,1] & \text{if} \quad y = 1/2, \\ 1 & \text{if} \quad y > 1/2, \\ 0 & \text{if} \quad y < 1/2 \end{cases}$$

- Similarly, by considering the utility to p2

$$\Rightarrow \quad y^* = \begin{cases} [0,1] & \text{if} \quad x = 1/2, \\ 1 & \text{if} \quad x > 1/2, \\ 0 & \text{if} \quad x < 1/2 \end{cases}$$

Matching Pennies (continued)

- Visualize best response strategies of two players.



- $x^* = y^* = 1/2$ is the unique NE.
- Zero-Sum Game
  - Only two players, P1 and P2.
  - Utility to p1 = - (Utility to P2).
  - So the game can be represented by only one entry in each cell, i.e., the matrix represents the utility to p1.

	Head	Tail
Head	$+1$	-1
Tail	-1	$+1$

Prisoner's Dilemma

- Two prisoners; two strategies: C: Confess; D: Deny.

P1/P2	C	D
C	$(-3,-3)$	$(0,-5)$
D	$(-5,0)$	$(-1,-1)$

- (C, C) is the only pure NE.
- x= p(p1 confess); y = p(p2 confess)
  Utility to p1: $-3xy - 5(1-x)y - (1-x)(1-y) = x(1+y) + \cdots$
- $x^* = 1$. Similarly $y^* = 1$. $\Rightarrow$ (C, C) is the unique NE.
- But (D, D) is the cooperative optimal solution. But is not a NE.

Comments for NE

We have seen examples, where

- there is a unique NE, which is a pure NE.
- there is a unique mixed NE.
- there exist multiple NEs with different utilities.

Two basic approaches to find NE

- compute the complete best response mapping for each agent. Find the intersection (graphically or otherwise).
- Fixed a strategy profile $(p_1, p_2, \cdots, p_n)$, check if any agent has a profitable deviation.

Dominant and Dominated Strategy

Dominant Strategy

A strategy p_i is called a dominant strategy for agent i if

$$u_i(p_i, p_{-i}) \geq u_i(p_i', p_{-i}) \qquad \forall p_i', p_{-i}$$

Dominant Strategy Equilibrium

A strategy profile $(p_1, \cdots, p_n)$ is called dominant strategy equilibrium if p_i is a dominant strategy for each agent i

- Dominant Strategy Equilibrium (DSE) is stronger than Nash Equilibrium
- In general, DSE may not exist in some game, but (mixed) NE always exists from Nash's Theorem.

Example for DSE

- Partnership Game

P1 / P2	Work Hard	Be Lazy
Work Hard	(10,10)	(-5, 5)
Be Lazy	(5,-5)	(0,0)

In this game, DSE does not exist, but pure NE and mixed NE exist.

- Prisoner's dilemma

Prisoner 1 / Prisoner 2	Confess	Deny
Confess	(-3,-3)	(0, -5)
Deny	(-5,0)	(-1,-1)

(C, C) is DSE and is also a NE. Action C is as good as any other strategy for each player.

Dominated Strategy

- A strategy p_i is said to be **strictly dominanted** for agent i if $\exists p_i'$ s.t.,

$$u_i(p_i', p_{-i}) > u_i(p_i, p_{-i}), \qquad \forall p_{-i}.$$

- A strategy p_i is said to be **weakly dominanted** for agent i if $\exists p_i'$ s.t.,

$$u_i(p_i', p_{-i}) \geq u_i(p_i, p_{-i}), \quad \forall p_{-i},$$
$$u_i(p_i', p_{-i}) > u_i(p_i, p_{-i}), \quad \exists p_{-i}.$$

- No agent would play a strictly dominated strategy, and thus we can remove such a strategy when analyzing a game.
- New option "Suicide" for P1.

Prisoner 1 / Prisoner 2	Confess	Deny
Confess	(-3,-3)	(0, -5)
Deny	(-5,0)	(-1,-1)
Suicide	(-100, -3)	(-100,0)

P1 is never going to play S, so that we can eliminate this row from the game.

Traffic Light Game

P1 / P2	Go	Yield
Go	(-10,-10)	(5, 0)
Yield	(0,5)	(-1,-1)

- **Pure NE**: (G, Y) and (Y, G).
- **Mixed NE**: $p = P(\text{agent 1 plays "Go"})$, $p^* = q^* = \frac{3}{8}$, the expected utility to P1 is $-\frac{15}{32}$. Homework.
- Suppose there is a traffic light, which correlates the actions of the agents:

$$P((G, Y)) = 0.5, P((Y, G)) = 0.5.$$

- Is there any incentive for P1 to deviate? No!
- If agent P1 is told to play G, it knows that P2 is playing Y, and G is the $BP(Y)$ of P1.
  Similarly, when p1 is told to play Y, it knows P2 is playing G, and Y is the $BP(G)$ of P1.
- Expected Utility to P1 is $\frac{5}{2}$.

Traffic Light Game (continued)

- Consider a different traffic light.
  $P((G, Y)) = 0.55$, $P((Y, G)) = 0.4$ and $P((Y, Y)) = 0.05$.
- If P1 is told to play G, it knows P2 is playing Y, and P1 $BP(Y)$ is G.
- It gets a little more complicated if P1 is told to play Y. Using Bayes's rule, P1 can infer the probability that P2 of playing Y or G.

$$P(x_2 = Y|x_1 = Y) = \frac{P(Y, Y)}{P(x_1 = Y)} = \frac{0.05}{0.45} = \frac{1}{9}.$$

$$P(x_2 = G|x_1 = Y) = \frac{P(G, Y)}{P(x_1 = Y)} = \frac{0.4}{0.45} = \frac{8}{9}.$$

- If P1 sticks to Y suggested by the traffic light, its expected utility is: $\frac{1}{9} \times 0 + \frac{8}{9} \times 1 = \frac{8}{9}$.
- On the other hand, if P1 decide to play G when suggested to play Y, its expected utility under this decision is $\frac{1}{9} \times 5 + \frac{8}{9} \times -1 = -\frac{3}{9}$.
- It is not profitable to unilaterally deviate from the suggestions of the traffic light.

Correlated Equilibrium

Correlated Equilibrium

Let $p^*(\mathbf{x})$ be a (join) probability distribution over $\mathbf{x} \in X_1 \times X_2 \times \cdots \times X_n$. The correlated mixed strategy $p^*(\mathbf{x})$ is a Correlated Equilibrium (CE) if

$$\sum_{\mathbf{x}_{-i}} p^*(\mathbf{x}_{-i}|x_i) \times u_i(x_i, \mathbf{x}_{-i}) \geq \sum_{\mathbf{x}_{-i}} p^*(\mathbf{x}_{-i}|x_i) \times u_i(x_i', \mathbf{x}_{-i}) \quad \forall x_i, x_i'.$$

- Multiplying the above inequalities by $p^*(x_i)$ on both sides, the definition of CE can be equivalently written as

$$\sum_{\mathbf{x}_{-i}} p^*(x_i, \mathbf{x}_{-i}) \times u_i(x_i, \mathbf{x}_{-i}) \geq \sum_{\mathbf{x}_{-i}} p^*(x_i, \mathbf{x}_{-i}) \times u_i(x_i', \mathbf{x}_{-i}) \quad \forall x_i, x_i'.$$

- There can be many correlated equilibria. Correlated equilibrium defines a collection of linear inequalities in the variables $\{\mathbf{p}(\mathbf{x})\}_{\mathbf{x} \in X}$ along with $p(\mathbf{x}) \geq 0, \ \forall \mathbf{x} \in X$ and $\sum_{\mathbf{x} \in X} p(\mathbf{x}) = 1$.

Correlated Equilibrium (continued)

- Any p that satisfies the above linear inequalities/equality is a Correlated Equilibrium. One can define an objective, and solve the resulting LP to pick a CE that satisfies some objective.
- A mixed strategy NE is a special case of a correlated equilibrium. Thus, from Nash Theorem, a correlated equilibrium always exists.
- Interpretation: A trusted third party (traffic light) samples a strategy profile $\mathbf{x}$ from $p^*(\mathbf{x})$. The trusted third party privately suggests the strategy x_i to agent i, who can follow x_i or not. CE guarantees that all agents would follow the suggestion.

A Hierarchy of Equilibrium Concepts

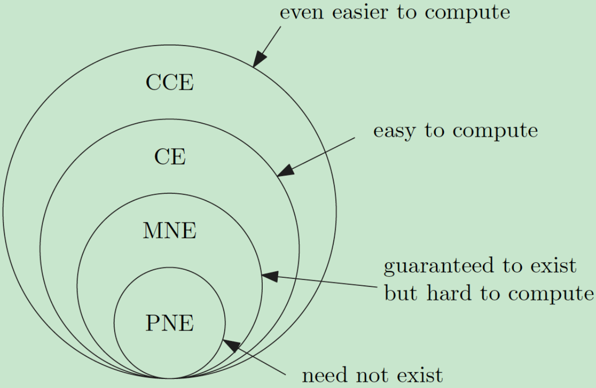


Figure 13.1: A hierarchy of equilibrium concepts: pure Nash equilibria (PNE), mixed Nash equilibria (MNE), correlated equilibria (CE), and coarse correlated equilibria (CCE).

Thanks!