SHANGHAI JIAO TONG UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

Game Theory with Computer Science Applications

Homework 2

May 13, 2021

Problem 1. Let X and Y be subsets of some vector space. Let $f(x, y)$ be a function from $X \times Y$ to $\mathbb{R}$. Show that

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \le \inf_{y \in Y} \sup_{x \in X} f(x, y).$$

Problem 2. Find a saddle point and the value of the following zero-sum game:

$$\begin{pmatrix} 4 & 3 & 1 & 4 \\ 2 & 5 & 6 & 3 \\ 1 & 0 & 7 & 0 \end{pmatrix}$$

Please show all the steps you used in obtaining the saddle point, such as the relevant LPs. If you used a computer program, please attach a copy of the program.

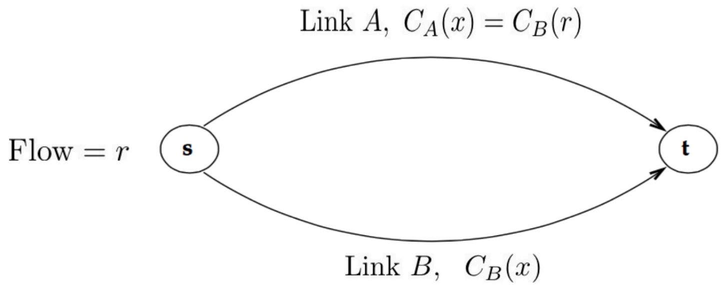
Problem 3. Repeat Problem 2 for the following matrix:

$$\begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}$$

Problem 4. Suppose n random variables Z_i's follow i.i.d. normal distribution, i.e., $Z_i \sim N(0,1)$, $\forall i = \{1, 2, cdots, n\}$, Prove the following two statements.

$$(1) \quad lim_{n \to \infty} \mathbb{E}\left(\frac{min_i Z_i}{\sqrt{2lnn}}\right) = 1.$$

$$(2) \quad \mathbb{E}(max_i Z_i) \le \sqrt{2lnn}.$$



Problem 5. Consider the following Pigou network: show that the price of anarchy (POA) when $C_B(x)$ is of the form $ax^2 + bx + c, a, b, c > 0$ is upper bounded by $\frac{3\sqrt{3}}{3\sqrt{3}-2}$.

Problem 6. Consider an undirected graph with n nodes and one player at each node. Player i selects a label $x_i \in \{-1, 1\}$ for node i and incurs a reward

$$ui(x_i; x_{-i}) = \sum_{j \neq i} w_{ij} \times x_i \times x_j,$$

where w_{ij} is the weight of the edge between i and j. Note that w_{ij}'s can be positive or negative or zero (when there's no edge between i and j), and each w_{ij} represents the desire of each node to havethe same label as its neighbors or not. Show that there exists a pure strategy Nash equilibrium $(x_1^*, x_2^*, \cdots, x_n^*)$ for this game. Hint: There is an exact potential function for this game.

Problem 7. Consider a player whose goal is to predict the bits of an infinite binary sequence $B_1, B_2, B_3, \cdots$, with $B_i \in \{0, 1\}$. Assume that

(1) The player has access to n experts. Each expert i provides a prediction bit of B_t before it is revealed based on $H_t = \{B_1, \cdots, B_{t-1}, b_{1,1}, \cdots, b_{1,t-1}, \cdots, b_{n,1}, \cdots, b_{n,t-1}\}$.

(2) The player predicts B_t based on $\{H_t, b_{1,t}, \cdots, b_{n,t}\}$.

(3) There exists at least one expert which predicts the entire sequence correctly.

Show that there exists an algorithm for the player such that it makes no more than $\lceil log2(n)e \rceil$ mistakes. *Hint: Consider a majority vote among experts who have been correct so far.*