SHANGHAI JIAO TONG UNIVERSITY
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING

Game Theory with Computer Science Applications

Homework 1

April 12, 2021

Problem 1. Show that the two-player game illustrated in the following has a unique equilibrium. (Hint: Show that it has a unique pure-strategy equilibrium; then show that player 1, say, cannot put positive weight on both U and M; then show that player 1, say, cannot put positive weight on both U and D, but not on M, for instance.)

$$\begin{pmatrix} & L & M & R \\ U & 1,-2, & -2,1 & 0,0 \\ M & -2,1 & 1,-2 & 0,0 \\ D & 0,0 & 0,0 & 1,1 \end{pmatrix}$$

Problem 2. [Cournot Competition] Consider two companies, say company 1 and company 2, which produce identical products. In the Cournot model of competition, companies decide the amount they produce and the market determines a price depending on the total amounts of the products available in the market. The price is higher if the amount of the product is smaller. Let $a_i (i = 1,2) \in [0,\infty)$ denote the amount of the product produced by company i. Assume that producing one unit of the product costs each company \$1, and the sales price per unit of the product is determined as $[2-(a_1+a_2)]^+$. Thus, the payoffs of company 1 and company 2 are given by

$$u_1(a_1,a_2) = a_1[2-(a_1+a_2)]^+ - a_1$$

$$u_2(a_1,a_2) = a_2[2-(a_1+a_2)]^+ - a_2,$$

respectively. Fine a pure Nash Equilibrium for this game.

Problem 3. [Bertrand Competition] The Bertrand model is an alternative to the Cournot model of competition. In the Bertrand model, again we consider two companies only, but now each company sets a price and the demand for the product is a function of the lower of the two companies' prices. More precisely, each company i sets a price p_i for the product. The demand for the product is a function of the prices as follows: if company i sets its price lower than that of the other company, i.e., $p_i < p_{-i}$, the demand for the product of company i is given by $f(p_i)$ units, and the demand for the product of the other company is zero. If $p_i = p_{-i}$, then the demand is $f(p_i)/2$ for both companies. Let c_i be the cost for company i to product one unit of the product. Then, the payoff for company i is given by

$$u_i(p_i, p_{-i}) = \begin{cases} f(p_i)(p_i - c_i) & \text{if} \quad p_i < p_{-i}, \\ f(p_i)(p_i - c_i)/2 & \text{if} \quad p_i = p_{-i} \\ 0 & \text{otherwise,} \end{cases}$$

show that when $c_1 = c_2 = c$, $p_1 = p_2 = c$ is the unique NE.

Problem 4. Find all the NE of the following two-person nonzero-sum game

	b_1	b_2	b_3	b_4
a_1	$(-2,2)$	$(0,-4)$	$(11,-5)$	$(5,-6)$
a_2	$(-4,0)$	$(-1,-1)$	$(11,-2)$	$(4,-3)$
a_3	$(-5,3)$	$(-5,2)$	$(10,0)$	$(3,1)$
a_4	$(-6,2)$	$(-7,1)$	$(1,0)$	$(2,3)$

Problem 5. Consider the following nonzero game. Let (x^*, y^*) and $(\hat{x}, \hat{y})$ be two mixed strategy Nash equilibria of this game. Show that $(x^*, \hat{y})$ and $(\hat{x}, y^*)$ are also Nash equilibria. (Hint: Consider the sum of the payoffs of the two players.)

	L	R
U	$(4,-2)$	$(-3,5)$
D	$(10,-8)$	$(0,2)$

Problem 6. Prove Farka's Lemma. Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^{m\times 1}$. Then exactly one of the following two conditions holds:
(1) $\exists x \in \mathbb{R}^{n\times 1}$ such that $AX = b$, $x \geq 0$;
(2) $\exists y \in \mathbb{R}^{1\times m}$ such that $A^T y \geq 0$, $y^T b < 0$;